On generalized Nambu mechanics

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A geometric formulation of a generalization of Nambu mechanics is proposed. This formulation is carried out, wherever possible, in analogy with that of Hamiltonian systems. In this formulation, a strictly nondegenerate constant 3-form is attached to a 3n- dimensional phase space. Time evolution is governed by two Nambu functions. A Poisson bracket of 2-forms is introduced, which provides a Lie-algebra structure on the space of 2-forms. This formalism is shown to provide a suitable framework for the descrip tion of non-integrable fluid flow such as the Arter flow, the Chandrashekhar flow and of the coupled rigid bodies.

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I. INTRODUCTION

In 1973, Nambu proposed a generalization of Hamiltonian mechanics by considering systems which obey Liouville theorem in 3 dimensional phase space [1]. In this formalism, the points of the phase space were labeled by a canonical triplet $\tilde{r} = (x, y, z)$. A pair of Hamiltonian like functions H_1, H_2, (which we call Nambu functions hereafter), on this phase space were introduced. In terms of these functions the equations of motion were written as

$$\frac{d\tilde{r}}{dt} = \tilde{\nabla} H_1 \times \tilde{\nabla} H_2 \tag{1}$$

Nambu also defined a generalization of Poisson bracket on this new phase space by

$$\{F, H_1, H_2\} = \tilde{\nabla} F \cdot (\tilde{\nabla} H_1 \times \tilde{\nabla} H_2) \tag{2}$$

An attempt was made by him to find a quantized version of the formalism, but he succeeded just partially, since the correspondence between the classical and quantum version is largely lost [1].

The possibility of embedding the dynamics of a Nambu triplet in a four dimensional canonical phase space formalism was proved in [2,3] but such an embedding is local and non-unique.

An algebraic approach, which was suitable for quantization was developed in [4,5] where a generalization of the Nambu bracket was postulated. In this approach a rather rigid consistency condition, called the Fundamental identity, which is a generalization of the Jacobi identity for Poisson brackets, was introduced. The algebraic approach, no doubt, is quite elegant but is too restri ctive; in the sense that the dynamics on a n-dimensional manifold is determined by $(n-1)$ functions $H_1, \ldots, H_{n-1}$ which are integrals of motions. Due to these large number of integrals of motion, this formalism is not suitable for the formulati on of non-integrable or chaotic systems.

Geometric formulation of Hamiltonian mechanics has revealed several deep insights. One would expect that a similar insight would emerge from geometric formulation of Nambu systems. Such possibilities were first examined by Estabrook [6] and more recently by Fecko [7].

Recall that in the formulation of Hamiltonian systems, there exist several equivalent ways [8] of endowing an even dimensional manifold M^{2n}, with a symplectic structure. Two of the prominent ways are:

1. attach a closed non-degenerate 2-form $\omega^{(2)}$.
2. attach a bracket on the class of C^∞ functions on M^{2n} with properties of bilinearity, skew-symmetry, Leibnitz rule, Jacobi identity and non-degeneracy (i.e., Poisson structure together with non-degeneracy).

Both types of approaches have been tried with Nambu systems [1–7]

- As mentioned above, it was found that the Algebraic approach, starting with bracket of functions, required the introduction of a rather rigid condition in the form of Fundamental Identity [5,9], for consistency.

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- On the other hand the geometric analysis [6,7] led to the conclusion, that a volume form was impossible to obtain from a 3-form, except in the most trivial cases.

The complications mentioned above arise in both cases because the volume preservation is a very stringent requirement. In light of these findings we propose in this paper, that one need not impose volume preservation for constructing geometric formalism of Nambu systems.

In this paper, we restrict ourselves to what we call a Nambu system of order 3. Such a system has a 3n dimensional phase space and two Nambu functions. We introduce a Nambu manifold as a 3n-dimensional manifold M^{3n} together with a constant 3-form $\omega^{(3)}$ which is strictly non-degenerate (The notion of non-degeneracy of 2-forms requires modification. This modified notion is called strict non-degeneracy). There is a natural generalization of Darboux basis corresponding to the Hamiltonian systems. Equati ons of motion (Nambu equations) are introduced in terms of two Nambu functions which are analogous to Hamiltonian function in Hamiltonian dynamics. A novel feature of the present paper is – there is a natural way of introducing the Nambu-Poisson bracket of 2-forms. The form $\omega^{(3)}$ is preserved in the present approach which may be compared with the preservation of $\omega^{(2)}$ for a Hamiltonian system. In Hamiltonian dynamics all powers of $\omega^{(2)}$ are canonical invariants. However, in the present formalism, since $\omega^{(3)}$ is a form of an odd order, no conclusions about canonical invariants can be obtained from preservation of $\omega^{(3)}$ itself.

The real justification for such a generalization can emerge from applications to realistic physical systems and from better algorithmic strategies. We demonstrate that the non-integrable Arter flow and the Chandrashekhar flow describing Rayleigh -Benard convective motion with rotation, can suitably be described in our framework. We further note that the algorithmic strategy developed in citeHandK can now be identified as a generalization in the our framework of symplectic integration corresponding to Hamiltonian framework.

In section II we have developed the geometric formulation. In section II A, The notion of strict non-degeneracy of 3-forms is introduced. The notion of Nambu vector space is defined using strict non-degeneracy. The existence of a Darboux like basis is proved. This is followed by the notion of a Nambu map. In section II B a Nambu manifold of order 3 and canonical transformations on this manifold are defined. Further a correspondence between 2-forms and vectors is established. This is followed by a discussion of conditions under which the 3-form $\omega^{(3)}$ is preserved. In section II C the Nambu systems are defined. A Nambu vector field corresponding to two Nambu functions is introduced. It is proved that the phase flow preserves the Nambu structure. In section II D a bracket of 2-forms is introduced. This bracket provides a Lie algebra structure on the space of 2-forms.

In section III concrete applications of this framework to the examples of coupled rigid bodies and to the fluid flows are described.

II. GEOMETRIC FORMULATION OF NAMBU SYSTEMS

We begin by recalling the essential features of the Hamiltonian formalism. The phase space has the structure of a smooth manifold M. A closed, nondegenerate 2-form, viz the symplectic 2-form $\omega^{(2)}$, is attached to this manifold. The non-degeneracy of $\omega^{(2)}$ imposes the condition that M be even dimensional. Canonical transformations are those transformations under which the 2-form $\omega^{(2)}$ remains invariant. Use of $\omega^{(2)}$ allows us to establish an isomorphism between 1-forms and vector fields. The time evolution is governed by the Hamiltonian vector field X_H which is just the vector field associated with the 1-form dH, where H is a smooth function on M.

Alternatively one can introduce Poisson brackets on the space of C^∞ functions on M. Poisson bracket together with non-degeneracy condition induces a symplectic structure.

In the present paper, wherever possible, we develop the framework of Nambu systems in analogy with that of Hamiltonian systems.

A. Nambu vector space

In this section we define the Nambu vector space which is analogous to the symplectic vector space in Hamiltonian mechanics. Nambu vector space is a vector space with a strictly non-degenerate 3-form. We prove that in such a space there exists a preferred choice of basis which we call as Nambu-Darboux basis.

Definition II.1 *(Nondegenerate form) : Let E be a finite dimensional vector space and let $\omega^{(3)}$ be a 3-form on E i.e.,*

$$\omega^{(3)} : E \times E \times E \to \mathbb{R}$$

the form $\omega^{(3)}$ is called a nondegenerate form if

$$\forall \ non \ \ zero \ \ e_1 \in E \ \exists \ \ e_2, e_3 \in E \ \ such \ \ that \ \ \omega^{(3)}(e_1, e_2, e_3) \neq 0$$

Remarks:

1. In three dimensions every non zero 3-form is non-degenerate.
2. If the dimension of the vector space is less than three, then one cannot have an anti-symmetric, non-degenerate 3-form.
3. A nondegenerate 3-form allows us to define an analog of orthogonal complement as follows.

Definition II.2 *(Nambu complement) : Let E be an m dimensional vector space with $m \geq 3$. Let $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E. Let us choose $e_1, e_2, e_3 \in E$ such that $\omega^{(3)}(e_1, e_2, e_3) \neq 0$. Let $P_1 = Span(e_1, e_2, e_3)$, then the Nambu complement of P_1 is defined as*

$$P_1^{\perp_E} = \{z \in E \ | \ \omega^{(3)}(z, z_1, z_2) = 0 \ \ \forall \ \ z_1, z_2 \in P_1\}$$

Proposition II.1 *Let E be an m dimensional vector space with $m \geq 3$. Let $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E. Let us choose $e_1, e_2, e_3 \in E$ such that $\omega^{(3)}(e_1, e_2, e_3) \neq 0$. We further choose e_2, e_3 such that $\omega^{(3)}(e_1, e_2, e_3) = 1$. Let $P_1 = Span(e_1, e_2, e_3)$, then $E = P_1 \oplus P_1^{\perp_E}$*

Proof: We write $\forall \ x \in E$

$$x' = x - \omega^{(3)}(x, e_2, e_3)e_1 - \omega^{(3)}(x, e_3, e_1)e_2 - \omega^{(3)}(x, e_1, e_2)e_3$$

It is easy to see that $x' \in P_1^{\perp_E}$. From definition of $P_1^{\perp_E}$ it follows that $P_1 \cap P_1^{\perp_E} = \{0\}$. Hence $E = P_1 \oplus P_1^{\perp_E}$

□

Definition II.3 *(Strictly nondegenerate form) : Let E be an m dimensional vector space and $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E, the $\omega^{(3)}$ is called strictly non-degenerate if for each non zero $e_1 \in E$ $\exists$ a two dimensional subspace $E_1 \subset E$ such that*

1. *$\omega^{(3)}(e_1, x_1, x_2) \neq 0$ $\forall$ linearly independent $\{e_1, x_1, x_2\}$ where $x_1, x_2 \in F_1$ and $F_1 = Span(e_1 + E_1)$.*
2. *$\omega^{(3)}(e_1, z_1, z_2) = 0$ $\forall$ $z_1, z_2 \in F_1^{\perp_E}$.*

Remarks:

1. In three dimensional space every non-degenerate form is strictly non-degenerate.

Definition II.4 *(Nambu vector space) : Let E be a finite dimensional vector space and $\omega^{(3)}$ be a completely anti-symmetric and strictly nondegenerate 3-form on E then the pair $(E, \omega^{(3)})$ is called a Nambu vector space.*

Recall that the rank of 2-form is defined as the rank of its matrix representation. We now introduce the notion of rank for the anti-symmetric 3-forms.

Definition II.5 *(Rank of $\omega^{(3)}$) : Let E be a finite dimensional vector space and $\omega^{(3)}$ be a completely anti-symmetric 3-form, then the rank of $\omega^{(3)}$ is defined as*

$$\sup_{P \subset E} \{d \mid d = dim P, \ (P, \omega^{(3)}|_P) \ is \ \ a \ \ Nambu \ \ vector \ \ space \ \}$$

Remarks:

1. The following proposition gives the prescription to construct the Nambu-Darboux basis.

Proposition II.2 *Let E be an m dimensional vector space. Let $\omega^{(3)}$ be a 3-form of rank m on E, then $m = 3n$ for a unique integer n. Further there is an ordered basis $\{e_i\}, i = 1, \dots, m$ with the corresponding dual basis $\{\alpha^i\}, i = 1, \dots, m$, such that*

$$\omega^{(3)} = \sum_{i=0}^{n-1} \alpha^{3i+1} \wedge \alpha^{3i+2} \wedge \alpha^{3i+3} \quad if \;\; n > 0$$
$$\omega^{(3)} = 0 \quad otherwise$$

Proof: The rank of $\omega^{(3)}$ is m, implies that $(E, \omega^{(3)})$ is a Nambu vector space. One can assume that $m \geq 3$, for otherwise the proposition is trivial with $n = 0$. Let $e_1 \in E$ and let $E_1 \subset E$ be a 2-dimensional subspace such that $\omega^{(3)}(e_1, e_2, e_3) \neq 0 \; \forall$ linearly independent $\{e_1, e_2, e_3\}$ where $e_2, e_3 \in P_1$ and $P_1 = Span(e_1 + E_1)$. Let e_2, e_3 be a basis of E_1, then e_1, e_2, e_3 is a basis of P_1. Thus $(P_1, \omega^{(3)}|_{P_1})$ is three dimensional Nambu vector space. It follows that one can write $\omega^{(3)}|_{P_1}$ in basis $\alpha^1, \alpha^2, \alpha^3$ dual to e_1, e_2, e_3 as

$$\omega^{(3)}|_{P_1} = \alpha^1 \wedge \alpha^2 \wedge \alpha^3$$

If $m = 3$ then $P_1 = E$ and the proposition is proved with $n = 1$. Hence we assume $m > 3$. We denote $Q_1 = P_1^{\perp_E}$. The dimension of Q_1 is $m - 3$. Since the vector space E is Nambu vector space it follows $dim(Q_1) \geq 3$. Let $e_4 \in Q_1 \subset E \Rightarrow \exists \; e_5, e_6 \in E$ such that $\omega^{(3)}(e_4, e_5, e_6) \neq 0 \; \forall$ linearly independent $\{e_4, e_5, e_6\}$ where $e_5, e_6 \in P_2$ and $P_2 = Span(e_4, e_5, e_6)$. By definition of Q_1 it follow s that $e_5, e_6 \notin P_1$. We choose $e_5, e_6 \in P_1^{\perp_E} = Q_1$. It follows from the strict non-degeneracy of $\omega^{(3)}$ in E and the facts $P_2^{\perp_{Q_1}} \subset P_2^{\perp_E}$ and $P_2 \subset Q_1$ that $(Q_1, \omega^{(3)}|_{Q_1})$ is a Nambu vector space of dimension $m - 3$.

repeated application of the above argument on Q_1 in place of E and so on yields

$$E = P_1 \oplus \cdots \oplus P_n$$

We stop the recursion when $dim(Q_n) = 0$. Using the strict non-degeneracy of $\omega^{(3)}$ we get

$$\begin{aligned}\omega^{(3)} &= \omega^{(3)}|_{P_1} + \cdots + \omega^{(3)}|_{P_n} \\ &= \sum_{i=0}^{n-1} \alpha^{3i+1} \wedge \alpha^{3i+2} \wedge \alpha^{3i+3}\end{aligned}$$

□

Definition II.6 *(Nambu mappings) : If (E, ω) and (F, ρ) are two Nambu vector spaces and $f : E \to F$ is a linear map such that the pullback $f^*\rho = \omega$, then f is called a Nambu mapping.*

Proposition II.3 *Let (E, ω) and (F, ρ) be two Nambu vector spaces of dimension $3n$ and let a linear map $f : E \to F$ be Nambu mapping then f is an isomorphism on the vector spaces.*

Proof: Let if possible, f be singular. Then there exists $x \in E$ and $x \neq 0$ such that $fx = 0$. But since f is a Nambu mapping one can write

$$\rho(f(x), f(y), f(z)) = \omega(x, y, z)$$

where $y, z \in E$ are so chosen that $\omega(x, y, z) \neq 0$. Which leads to contradiction. Hence f is an isomorphism.

□

Proposition II.4 *Let $(E, \omega^{(3)})$ be a Nambu vector space of dimension $3n$, then the set of Nambu mappings from E to E forms a group under composition.*

Proof: Now let f and g are Nambu mappings.

$$(f \circ g)^* \omega^{(3)} = g^* \circ f^* \omega^{(3)} = g^* \omega^{(3)} = \omega^{(3)}$$

and

$$(f^{-1})^* \omega^{(3)} = (f^*)^{-1} \omega^{(3)} = \omega^{(3)}$$

□

B. Nambu manifold

In analogy with the notion of symplectic manifold we now introduce Nambu manifold.

Definition II.7 *(Nambu Manifold) : Let M^{3n} be a $3n$-dimensional C^{∞} manifold and let $\omega^{(3)}$ be a 3-form field on M^{3n} such that $\omega^{(3)}$ is completely anti-symmetric, constant (i.e., a constant section on the bundle of 3-forms) and strictly nondegenerate at every point of M^{3n} then the pair $(M^{3n}, \omega^{(3)})$ is called a Nambu manifold.*

Remarks:

1. In the case of Hamiltonian systems the form $\omega^{(2)}$ is assumed to be closed, here $\omega^{(3)}$ is assumed to be a constant form. The condition of closedness allows many 3-forms which in general may not be consistent with the non-degeneracy conditio n.

Theorem II.1 *(Nambu-Darboux theorem) : Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold then at every point $p \in M^{3n}$, there is a chart (U, ϕ) in which $\omega^{(3)}$ is written as*

$$\omega^{(3)}|_U = \sum_{i=0}^{n-1} dx_{3i+1} \wedge dx_{3i+2} \wedge dx_{3i+3}$$

where $(x_1, x_2, x_3, \ldots, x_{3(n-1)+1}, x_{3(n-1)+2}, x_{3(n-1)+3})$ are local coordinates on U described by ϕ.

Proof: Proof follows from Proposition II.2.

□

The coordinates described in theorem II.1 will be called Nambu-Darboux coordinates hereafter. We use these coordinates in the remaining parts of the paper.

Definition II.8 *(Canonical transformation) : Let $(M^{3n}, \omega^{(3)})$ and $(N^{3n}, \rho^{(3)})$ be Nambu manifolds. A C^{∞} mapping $F : M^{3n} \to N^{3n}$ is called canonical transformation if $F^*\rho^{(3)} = \omega^{(3)}$.*

Let $\mathcal{T}_k^0(M^{3n})$ denote a bundle of k-forms on M^{3n}, $\Omega_k^0(M^{3n})$ denote the space of k-form fields on M^{3n} and $\mathcal{X}(M^{3n})$ denote the space of vector fields on M^{3n}. Now for a given vector field X on M^{3n} we denote

$$i_X : \Omega_k^0(M^{3n}) \to \Omega_{k-1}^0(M^{3n})$$

as inner product of X with k-form or contraction of a k-form by X given by

$$(i_X \eta^{(k)})(a_1, \ldots, a_{k-1}) = \eta^{(k)}(X, a_1, \ldots, a_{k-1})$$

where $\eta^{(k)} \in \Omega_k^0(M^{3n})$ and $a_1, \ldots, a_{k-1} \in \mathcal{X}(M^{3n})$

We define now the analogs of raising and lowering operations. The map $\flat : \mathcal{X}(M^{3n}) \to \Omega_2^0(M^{3n})$ is defined by $X \mapsto X^\flat = i_X \omega^{(3)}$. Whereas the map $\sharp : \Omega_2^0(M^{3n}) \to \mathcal{X}(M^{3n})$, is defined by the following prescription. Let α be a 2-form and α_{ij} be its components in Nambu-Darboux coordinates, then the components of $\alpha^\sharp$ are given by

$$\alpha^{\sharp\, 3i+p} = \frac{1}{2} \sum_{l,m=1}^{3} \varepsilon_{plm} \alpha_{3i+l \quad 3i+m} \tag{3}$$

where $0 \leq i \leq n-1$, $p = 1, 2, 3$ and ε_{plm} is the Levi-Cevità symbol.

Remarks:

1. It may appear that components of $\alpha^\sharp$ have been given a definition using a particular choice of coordinate system. The definition itself is actually coordinate free as shown in the Appendix.

2. In difference with the customary meaning of $\flat$ and $\sharp$ used in ordinary tensor analysis we note that in this paper $\flat$ maps a vector to 2-form and not to a 1-form, also $\sharp$ maps a 2-form to a vector. From the above definition it is clear that $(X^\flat)^\sharp = X$ but $(\alpha^\sharp)^\flat$ may not always yield the same α.

3. In fact consider $\mathcal{T}^0_{2_x}(M^{3n})$, the space of 2-forms at $x \in M^{3n}$. The $\sharp$ defines an equivalence relation on $\mathcal{T}^0_{2_x}(M^{3n})$ as follows. Let $\omega_1^{(2)}(x), \omega_2^{(2)}(x) \in \mathcal{T}^0_{2_x}(M^{3n})$. We say that $\omega_1^{(2)}(x) \sim \omega_2^{(2)}(x)$ if $(\omega_1^{(2)})^\sharp(x) = (\omega_2^{(2)})^\sharp(x)$. It is easy to see that $\sim$ is an equivalence relation. We define the equivalence classes $\mathcal{S}^0_{2_x}(M^{3n}) = \mathcal{T}^0_{2_x}(M^{3n})/\sim$.

   Let $\omega_1^{(2)}(x), \omega_2^{(2)}(x), \omega_3^{(2)}(x) \in \mathcal{T}^0_{2_x}(M^{3n})$ and the equivalence class be denoted by $[\ ]$ i.e., $\alpha_1(x) = [\omega_1^{(2)}(x)], \alpha_2(x) = [\omega_2^{(2)}(x)], \alpha_3(x) = [\omega_3^{(2)}(x)]$ where $\alpha_1(x), \alpha_2(x), \alpha_3(x) \in \mathcal{S}^0_{2_x}(M^{3n})$. The addition and scalar multiplication on $\mathcal{S}^0_{2_x}(M^{3n})$ are defined as follows.

$$\alpha_1(x) + \alpha_2(x) = [\omega_1^{(2)}(x) + \omega_2^{(2)}(x)]$$
$$\mu \cdot \alpha_1(x) = [\mu \cdot \omega_1^{(2)}(x)]$$

   where $\mu \in \mathbb{R}$. It is easy to see that $(\mathcal{S}^0_{2_x}(M^{3n}), +, \mathbb{R}, \cdot)$ forms a vector space and the dimension of this vector space is $3n$.

Now we investigate conditions under which the given Nambu form $\omega^{(3)}$ is invariant under the action of the vector field $\beta^\sharp$ associated with any 2-form β.

Proposition II.5 *Let $\beta^{(2)} \in \Omega^0_2(M^{3n})$ then $(\beta^{(2)^\sharp})^\flat \sim \beta^{(2)}$*

Proof: Proof follows from the fact that $(X^\flat)^\sharp = X \ \ \forall X \in \mathcal{X}(M^{3n})$

□

Theorem II.2 *Let $\beta^{(2)} \in \Omega^0_2(M^{3n})$, and f^t be a flow corresponding to $\beta^{(2)^\sharp}$, i.e., $f^t : M^{3n} \to M^{3n}$ such that*

$$\frac{d}{dt}\Big|_{t=0}(f^t x) = (\beta^{(2)^\sharp})x \ \ \forall x \in M^{3n}$$

Then the form $\omega^{(3)}$ is preserved under the action of $\beta^{(2)^\sharp}$ iff $d(\beta^{(2)^\sharp})^\flat = 0$. i.e., $f^{t*}\omega^{(3)} = \omega^{(3)}$ iff $d(\beta^{(2)^\sharp})^\flat = 0$

Proof:

$$\begin{aligned}\frac{d}{dt}(f^{t*}\omega^{(3)}) &= f^{t*}(L_{\beta^{(2)\sharp}}\omega^{(3)}) \\ &= f^{t*}(i_{\beta^{(2)\sharp}}d\omega^{(3)} + d(i_{\beta^{(2)\sharp}}\omega^{(3)})) \\ &= f^{t*}d(\beta^{(2)^\sharp})^\flat\end{aligned}$$

If $d(\beta^{(2)^\sharp})^\flat = 0 \Rightarrow \frac{d}{dt}(f^{t*}\omega^{(3)}) = 0$ and also if $\frac{d}{dt}(f^{t*}\omega^{(3)}) = 0 \Rightarrow d(\beta^{(2)^\sharp})^\flat = 0$

□

From the above theorem it follows that the vector field corresponding to a 2-form preserves Nambu structure if that 2-form is equivalent to a closed 2-form. By Poincaré lemma one can locally write the closed 2-form as $d\xi$ where ξ is 1-form. Witho ut loss of generality we choose $\xi = f_1 df_2$ where f_1 and f_2 are C^∞ functions on M^{3n}. So we can choose $(\beta^{(2)^\sharp})^\flat$ as $df_1 \wedge df_2$ and by proposition II.5 $(df_1 \wedge df_2) \sim \beta^{(2)}$.

C. Nambu system

Having introduced the relevant structure viz. the Nambu manifold, we now proceed with the discussion of time-evolution. The time-time-evolution is governed by a vector field obtained from two Nambu functions.

Definition II.9 *(Nambu vector field) : Let $\mathcal{H}_1$, $\mathcal{H}_2$ be real valued C^∞ functions (Nambu functions) on $(M^{3n}, \omega^{(3)})$ then N is called Nambu vector field corresponding to $\mathcal{H}_1, \mathcal{H}_2$ if*

$$N = (d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp$$

Definition II.10 *(Nambu system) : A four tuple $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ is called Nambu system.*

Henceforth we choose $d\mathcal{H}_1 \wedge d\mathcal{H}_2$ as the representative 2-form of the class $[d\mathcal{H}_1 \wedge d\mathcal{H}_2]$. [1]

Now for a given representative 2-form we have the freedom in the choice of $\mathcal{H}_1$ and $\mathcal{H}_2$. This freedom is discussed in [1] as gauge freedom in the choice of $\mathcal{H}_1$ and $\mathcal{H}_2$. [2]

Definition II.11 *(Nambu phase flow) : Let $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ be a Nambu system then the diffeomorphisms $g^t : M^{3n} \to M^{3n}$ satisfying*

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(g^t\mathbf{x}) &= (d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp \mathbf{x} \quad \forall\, \mathbf{x} \in M^{3n} \\ &= N\mathbf{x}\end{aligned}$$

is called Nambu phase flow.

From the properties of flow of a differentiable vector field it follows that g^t is one parameter group of diffeomorphisms.

Theorem II.3 *Nambu phase flow preserves Nambu structure. i.e.,*

$$g^{t*}\omega^{(3)} = \omega^{(3)}$$

Proof: Proof follows from Theorem II.2

□

Remarks:

1. Since the proof of theorem II.3 is valid for any time t provided the flow g^t exists, it is automatically valid for any time interval say from t_1 to t_2. Further the flow preserves the $\omega^{(3)}$, which implies that the map $g^t : M^{3n} \to M^{3n}$ is a canonical transformation. This leads to the interpretation, as in the case of Hamiltonian systems viz,: "*The history of a Nambu system is a gradual unfolding of successive canonical transformation.*" Such an o bservation is one of the crucial ingredients required for the development of symplectic integrators for Hamiltonian systems. The present observation therefore can be used for a similar algorithm for Nambu systems.

Proposition II.6 *Let $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ be a Nambu system, then $\mathcal{H}_1$ and $\mathcal{H}_2$ are constants of motion.*

Proof: We prove the result for $\mathcal{H}_1$ and proof is similar for $\mathcal{H}_2$

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_1 &= L_N\mathcal{H}_1 \\ &= i_N d\mathcal{H}_1 \\ &= d\mathcal{H}_1(N) \\ &= d\mathcal{H}_1((d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp)\end{aligned}$$

If we write the RHS in Nambu-Darboux coordinates using equations (3) we get RHS = 0.

□

D. Nambu bracket

We now define the analog of the Poisson bracket for 2-forms. This leads to the algebra of 2-forms. Further we also define the brackets for three functions in the conventional fashion [1].

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[1] We note that in the equivalence class of $d\mathcal{H}_1 \wedge d\mathcal{H}_2$ there are some 2-forms which are not closed and can not be expressed as $d\mathcal{H}_1 \wedge d\mathcal{H}_2$.

[2] Note that this freedom is different from the freedom in de finition of $\mathcal{S}_2^0$. This freedom is generalization of the freedom of additive constant in Hamiltonian dynamics.

Definition II.12 *(Nambu Poisson bracket) : Let $\omega_a^{(2)}$ and $\omega_b^{(2)}$ be 2-forms then the* *Nambu Poisson bracket* *is a map* $\{,\}: \Omega_2^0(M^{3n}) \times \Omega_2^0(M^{3n}) \to \Omega_2^0(M^{3n})$ *given by*

$$\{\omega_a^{(2)}, \omega_b^{(2)}\} = [\omega_a^{(2)\sharp}, \omega_b^{(2)\sharp}]^\flat$$

where $[,]$ is Lie bracket of vector fields.

that is, one writes

$$\{\omega_a^{(2)}, \omega_b^{(2)}\}(\xi,\eta) = (i_{[\omega_a^{(2)\sharp}, \omega_b^{(2)\sharp}]}\omega^{(3)})(\xi,\eta) \quad \forall\ \xi,\eta \in \mathcal{X}(M^{3n})$$

From the definition it clear that if $\omega_a^{(2)} \sim \omega_b^{(2)}$ then the $\{\omega_a^{(2)}, \omega_b^{(2)}\} = 0$ and if $\omega_a^{(2)} \sim \omega_{a'}^{(2)}$ and $\omega_b^{(2)} \sim \omega_{b'}^{(2)}$ then $\{\omega_a^{(2)}, \omega_b^{(2)}\} = \{\omega_{a'}^{(2)}, \omega_{b'}^{(2)}\}$

Proposition II.7 *Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold and $\alpha, \beta \in \Omega_2^0(M^{3n})$ then*

$$\{\alpha,\beta\} = L_{\alpha^\sharp}(\beta^\sharp)^\flat - L_{\beta^\sharp}(\alpha^\sharp)^\flat - d(i_{\alpha^\sharp} i_{\beta^\sharp}\omega^{(3)})$$

Proof:

$$\begin{aligned}
\{\alpha,\beta\} &= i_{[\alpha^\sharp,\beta^\sharp]}\omega^{(3)} \\
&= L_{\alpha^\sharp}(i_{\beta^\sharp}\omega^{(3)}) - i_{\beta^\sharp}(L_{\alpha^\sharp}\omega^{(3)}) \\
&= L_{\alpha^\sharp}(\beta^\sharp)^\flat - i_{\beta^\sharp} d(i_{\alpha^\sharp}\omega^{(3)}) \\
&= L_{\alpha^\sharp}(\beta^\sharp)^\flat - L_{\beta^\sharp}(\alpha^\sharp)^\flat - d(i_{\alpha^\sharp} i_{\beta^\sharp}\omega^{(3)})
\end{aligned}$$

□

Proposition II.8 *Let α and β be 2-forms. Further let $\alpha^\sharp$ be a Nambu vector field. Let $\alpha' = (\alpha^\sharp)^\flat$ and $\beta' = (\beta^\sharp)^\flat$ then*

$$\{\alpha,\beta\} = L_{\alpha'^\sharp}\beta'$$

Proof: Proof follows from proposition II.7

□

Proposition II.9 *The space $\Omega_2^0(M^{3n})$ forms a Lie Algebra with multiplication defined by the bracket i.e., If $\alpha,\beta,\gamma \in \Omega_2^0(M^{3n})$*

1. $\{\alpha+\gamma,\beta\} = \{\alpha,\beta\} + \{\gamma,\beta\}$ *and* $\{\alpha,\beta+\gamma\} = \{\alpha,\beta\} + \{\alpha,\gamma\}$
2. $\{\alpha,\alpha\} = 0$
3. $\{\alpha,\{\beta,\gamma\}\} + \{\beta,\{\gamma,\alpha\}\} + \{\gamma,\{\alpha,\beta\}\} = 0$

Proof: Follows from definition of bracket

□

Remarks:

1. In the Hamiltonian systems smooth functions on the phase space are considered as observables. To each such function f a natural vector field (viz X_f which is in correspondence with df) is attached. We note that it is really to the 1-form df that a vector field is attached (All functions differing by constants form an equivalence class producing identical df.)

   In view of the above discussion it is clear that in the Nambu framework 2-forms play a basic role. In this connection we point out the following facts:

   (a) Vector fields are naturally associated with 2-forms.

(b) The bracket of 2-forms provides the Lie Algebra structure. On the other hand bracket of functions introduces a non-associative structure as noted by [1,9].

Definition II.13 *(Nambu bracket for functions) : Consider a Nambu manifold* $(M^{3n}, \omega^{(3)})$ *and let* f, g, h *be* C^∞ *functions on* M^{3n} *then Nambu bracket for functions is given by*

$$\{f, g, h\} = L_{(dg\wedge dh)^\sharp} f = i_{(dg\wedge dh)^\sharp} df$$

By using equation (3) in Nambu-Darboux coordinates,

$$\{f, g, h\} = \sum_{i=0}^{n-1} \sum_{k,l,m=1}^{3} \varepsilon_{klm} \frac{\partial f}{\partial x_{3i+k}} \frac{\partial g}{\partial x_{3i+l}} \frac{\partial h}{\partial x_{3i+m}}$$

In three dimensions this is simply the definition of the bracket given by Nambu [1]

Proposition II.10 *Consider a Nambu system* $(M^{3n}, \omega^{(3)}, g, h)$. *Let* $f, g', h' \in C^\infty(M^{3n})$ *satisfying* $(dh \wedge dg) \sim (dh' \wedge dg')$ *and* $(df \wedge dg')^{\sharp^\flat} = (df \wedge dg')$, *then we have*

$$\{f, g, h\}dg' = i_{(dg\wedge dh)^\sharp} i_{(df\wedge dg')^\sharp} \omega^{(3)}$$

Proof:

$$\begin{aligned}\{f, g, h\}dg' &= (L_{(dg\wedge dh)^\sharp} f)dg' \\ &= (i_{(dg\wedge dh)^\sharp}(df \wedge dg')) \\ &= i_{(dg\wedge dh)^\sharp} i_{(df\wedge dg')^\sharp} \omega^{(3)}\end{aligned}$$

□

Following proposition gives the relation between the bracket of 2-forms and the bracket of functions.

Proposition II.11 *Let* $(M^{3n}, \omega^{(3)})$ *be a Nambu manifold and let* f, g, h_1, h_2 *be* C^∞ *functions satisfying* $(df \wedge dg)^{\sharp^\flat} = df \wedge dg$ *and* $(dh_1 \wedge dh_2)^{\sharp^\flat} = dh_1 \wedge dh_2$ *then*

$$\{dh_1 \wedge dh_2, df \wedge dg\} = d\{f, h_1, h_2\} \wedge dg + df \wedge d\{g, h_1, h_2\}$$

Proof: From Proposition II.7 we have

$$\{dh_1 \wedge dh_2, df \wedge dg\} = L_{(dh_1\wedge dh_2)^\sharp}(df \wedge dg)$$

□

Remark :

1. If a function f is an integral of motion then the Nambu bracket of function $\{f, \mathcal{H}_1, \mathcal{H}_2\}$ is zero and conversely. On the other hand if β is a 2-form such that $\{d\mathcal{H}_1 \wedge \mathcal{H}_2, \beta\} = 0$ then there exists a 2-form β' in the equivalenc e class of β which is an invariant of motion. Also by proposition II.11 these two statements are consistent.

III. APPLICATIONS OF NAMBU MECHANICS

The purpose of this section is to demonstrate that there are systems that can be described appropriately using the formalism developed here.

A. Fluid flows

It was known for a long time that two dimensional incompressible fluid flows can be studied using the two dimensional Hamiltonian framework. It was realized by Holm and Kimura [11] that for three dimensional integrable flows of incompressible fl uids in the Lagrangian picture, Nambu description is suitable. However the three dimensional Nambu system is not suitable as a framework for the formulation of non-integrable fluid flows. We now claim that this requirement can be fulfilled by an appropria te choice of $\mathcal{H}_1$, $\mathcal{H}_2$ in a $3n$ dimensional Nambu framework. Specifically we show below that the Arter and Chandrashekhar flows (describing Rayleigh-Benard convective motion) can be casted as flows on an invariant three dimensional subspace of a six d imensional Nambu system.

We begin with the case of Chandrashekhar flow.

1. Chandrashekhar flow

We now show that the Chandrashekhar flow admits the description in our framework as follows:
Consider a Nambu manifold $(\mathbb{R}^6, \omega^{(3)})$. Let $\{x, y, z, x', y', z'\}$ be the coordinates on $\mathbb{R}^6$ and $\omega^{(3)} = dx \wedge dy \wedge dz + dx' \wedge dy' \wedge dz'$ in these coordinates. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Nambu functions defined as

$$\mathcal{H}_1 = \log\Big(\frac{\sin(x)}{\sin(y)}\Big) - \log\Big(\frac{\sin(x')}{\sin(y')}\Big) - K^2\frac{\cos(x')}{\sin(x')}(y-y') - K^2\frac{\cos(y')}{\sin(y')}(x-x') \tag{4}$$

$$\mathcal{H}_2 = \sin(x)\sin(y)\sin(z) - \sin(x')\sin(y')\sin(z') + (y-y')A + (x-x')B \tag{5}$$

where K^2 is constant which will be identified later, and

$$A = \frac{K^2\cos(y')\cos(y)\sin^2(x)\sin(z)}{\cos(x)\sin(y') - K^2\sin(x)\cos(y')}$$

$$B = -\frac{K^2\cos(x')\cos(x)\sin^2(y)\sin(z)}{\cos(y)\sin(x') + K^2\sin(y)\cos(x')}$$

We will now show that the sub-space defined by

$$x = x', y = y', z = z' \tag{6}$$

is an invariant subspace[3] and also that the equations of motion for (x, y, z) (or equivalently (x', y', z')) are precisely the equations governing the Chandrashekhar flow [11].

We write the Nambu vector field for given functions $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\begin{aligned}\dot{x} = &-\sin(x)\cos(y)\cos(z) - K^2\cos(x')\sin(y)\cos(z)\frac{\sin(x)}{\sin(x')}\\ &+\Big(-\frac{\cos(y)}{\sin(y)} - K^2\frac{\cos(x')}{\sin(x')}\Big)\Big(\frac{\partial A}{\partial z}(y-y') + \frac{\partial B}{\partial z}(x-x')\Big)\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{y} = &-\cos(x)\sin(y)\cos(z) + K^2\cos(y')\sin(x)\cos(z)\frac{\sin(y)}{\sin(y')}\\ &-\Big(\frac{\cos(x)}{\sin(x)} - K^2\frac{\cos(y')}{\sin(y')}\Big)\Big(\frac{\partial A}{\partial z}(y-y') + \frac{\partial B}{\partial z}(x-x')\Big)\end{aligned} \tag{8}$$

$$\dot{z} = 2\cos(x)\cos(y)\sin(z) + \Big(\frac{\cos(x)}{\sin(x)} - K^2\frac{\cos(y')}{\sin(y')}\Big)\Big(\frac{\partial A}{\partial y}(y-y') + \frac{\partial B}{\partial y}(x-x')\Big)$$

---

[3] This subspace is not Nambu with respect to the $\omega^{(3)}$ defined above because the transformation from $\mathbb{R}^6$ to this subspace is not canonical

$$+\Big(\frac{\cos(y)}{\sin(y)}+K^2\frac{\cos(x')}{\sin(x')}\Big)\Big(\frac{\partial A}{\partial x}(y-y')+\frac{\partial B}{\partial x}(x-x')\Big) \tag{9}$$

$$\dot{x'}=-\sin(x')\cos(y')\cos(z')-K^2\cos(x')\sin(y')\cos(z')-K^2\frac{\sin(x')}{\sin(y')}\cos(z')(x-x') \tag{10}$$

$$\dot{y'}=-\cos(x')\sin(y')\cos(z')+K^2\sin(x')\cos(y')\cos(z')+K^2\frac{\sin(y')}{\sin(x')}\cos(z')(y-y') \tag{11}$$

$$\begin{aligned}\dot{z'}&=2\cos(x')\cos(y')\sin(z')+\mu+\nu+\frac{\partial\mathcal{H}_2}{\partial y'}K^2\mathrm{cosec}^2(x')(y-y')-K^2\frac{\partial\mathcal{H}_2}{\partial x'}\mathrm{cosec}^2(y')(x-x')\\&\quad+\alpha\frac{\partial A}{\partial y'}(y-y')+\alpha\frac{\partial B}{\partial y'}(x-x')-\beta\frac{\partial A}{\partial x'}(y-y')-\beta\frac{\partial B}{\partial y'}(x-x')\end{aligned} \tag{12}$$

where

$$\begin{aligned}\mu&=K^2\tfrac{\cos(y')}{\sin(y')}\Big[-\sin(x')\cos(y')\sin(z')+\sin(x)\cos(y)\sin(z)\\&\qquad\times\frac{\sin(x)}{\sin(x')}\Big(\frac{\cos(x')\sin(y')-K^2\sin(x')\cos(y')}{\cos(x)\sin(y')-K^2\sin(x)\cos(y')}\Big)\Big]\\ \nu&=K^2\tfrac{\cos(x')}{\sin(x')}\Big[\cos(x')\sin(y')\sin(z')-\cos(x)\sin(y)\sin(z)\\&\qquad\times\frac{\sin(y)}{\sin(y')}\Big(\frac{\sin(x')\cos(y')+K^2\cos(x')\sin(y')}{\cos(y)\sin(x')+K^2\sin(y)\cos(x')}\Big)\Big]\end{aligned}$$

$$\begin{aligned}\alpha&=-\frac{\cos(x')}{\sin(x')}+K^2\frac{\cos(y')}{\sin(y')}\\ \beta&=\frac{\cos(y')}{\sin(y')}+K^2\frac{\cos(x')}{\sin(x')}\end{aligned}$$

We now notice that on the subspace defined by equation (6) $\dot{x}=\dot{x'},\dot{y}=\dot{y'},\dot{z}=\dot{z'}$. Thus the subspace is invariant. Moreover in this subspace the vector field is

$$\begin{aligned}\dot{x}&=-\sin(x)\cos(y)\cos(z)-K^2\cos(x)\sin(y)\cos(z)\\ \dot{y}&=-\cos(x)\sin(y)\cos(z)+K^2\cos(y)\sin(x)\cos(z)\\ \dot{z}&=2\cos(x)\cos(y)\sin(z)\end{aligned}$$

which are precisely the equations governing the Chandrashekhar flow [11] where (x,y,z) are coordinates of the fluid particles in the Lagrangian description applicable to Rayleigh-Benard convection with rotation and K is constant proportional to rotation of the fluid [11].

2. Arter flow

Now we show that the Arter flow admits the description in our framework as follows:
Consider Nambu manifold $(\mathbb{R}^6,\omega^{(3)})$. Let $\{x,y,z,x',y',z'\}$ be the coordinates on $\mathbb{R}^6$ and $\omega^{(3)}=dx\wedge dy\wedge dz+dx'\wedge dy'\wedge dz'$ in this coordinate. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the N ambu functions defined as

$$\mathcal{H}_1=\log\Big(\frac{\sin(x)}{\sin(y)}\Big)-\log\Big(\frac{\sin(x')}{\sin(y')}\Big)-2b\frac{\cos(y)\cos(2z')}{\sin(x')\cos(z)}(x-x')+2b\frac{\cos(x)\cos(2z')}{\sin(y')\cos(z)}(y-y') \tag{13}$$

$$\mathcal{H}_2=\sin(x)\sin(y)\sin(z)-\sin(x')\sin(y')\sin(z')-(x-x')C+(y-y')D \tag{14}$$

where

$$\begin{aligned}C&=\frac{\cos(x)\sin(y)\Big((\cos^2(x)+\cos^2(y))\cos(2z)\sin(z)-(\cos(2x)\cos(2y))\sin(2z)\cos(z)\Big)}{\cos(2z)(\cos^2(x)-\cos^2(y))}\\ D&=\frac{\cos(y)\sin(x)\Big((\cos^2(x)+\cos^2(y))\cos(2z)\sin(z)-(\cos(2x)\cos(2y))\sin(2z)\cos(z)\Big)}{\cos(2z)(\cos^2(x)-\cos^2(y))}\end{aligned}$$

where b is constant. We have omitted the rather lengthy equations. As in the case of Chandrashekhar flow, the subspace defined by

$$x = x', y = y', z = z' \tag{15}$$

is an invariant subspace. The equations of motion for (x, y, z) (or equivalently (x', y', z')) in this subspace are

$$\begin{aligned}
\dot{x} &= -\sin(x)\cos(y)\cos(z) + b\cos(2x)\cos(2z) \\
\dot{y} &= -\cos(x)\sin(y)\cos(z) + b\cos(2y)\cos(2z) \\
\dot{z} &= 2\cos(x)\cos(y)\sin(z) - b(\cos(2x) + \cos(2y))\sin(2z)
\end{aligned}$$

which are precisely the equations governing the Arter flow [11] where (x, y, z) are coordinates of the fluid particles in the Lagrangian description applicable to Rayleigh-Benard convection with rotation [11].

B. Coupled rigid bodies

We now consider the simplest case of a coupling between two symmetric tops. The coupling introduced is proportional to the z component of angular momenta of each rotor (Such an idealized situation corresponds under certain assumptions to the case of two s ymmetric tops interacting with each other through magnetic moment coupling). The equations of motions for the angular momenta of the tops in their respective body coordinates are

$$\begin{aligned}
\dot{x_1} &= \frac{1}{I_{y_1} I_{z_1}}[y_1 z_1 (I_{z_1} - I_{y_1}) + I_{z_1} C_3 y_1 z_2] \\
\dot{y_1} &= -\frac{1}{I_{x_1} I_{z_1}}[x_1 z_1 (I_{z_1} - I_{x_1}) + I_{z_1} C_3 x_1 z_2] \\
\dot{z_1} &= 0 \\
\dot{x_2} &= \frac{1}{I_{y_2} I_{z_2}}[y_2 z_2 (I_{z_2} - I_{y_2}) + I_{z_2} C_3 y_2 z_1] \\
\dot{y_2} &= -\frac{1}{I_{x_2} I_{z_2}}[x_2 z_2 (I_{z_2} - I_{x_2}) + I_{z_2} C_3 x_2 z_1] \\
\dot{z_2} &= 0
\end{aligned}$$

These equations have the generalized Nambu form in the sense of this paper and can be obtained from the Nambu functions.

$$\begin{aligned}
\mathcal{H}_1 &= \frac{1}{2}({x_1}^2 + {y_1}^2 + {z_1}^2) + \frac{1}{2}({x_2}^2 + {y_2}^2 + {z_2}^2) \\
\mathcal{H}_2 &= \frac{1}{2}\Big(\frac{{x_1}^2}{I_{x_1}} + \frac{{y_1}^2}{I_{y_1}} + \frac{{z_1}^2}{I_{z_1}}\Big) + \frac{1}{2}\Big(\frac{{x_2}^2}{I_{x_2}} + \frac{{y_2}^2}{I_{y_2}} + \frac{{z_2}^2}{I_{z_2}}\Big) + C_3 z_1 z_2
\end{aligned}$$

It is obvious that the constant C_3 depends on the initial orientation in lab frame. In the absence of coupling the tops obey Euler equations individually. In above vector field the terms like $I_{z_1} C_3 y_1 z_2$ can be considered as effective externa l torque due to presence of another body. The important point to note is that this torque just changes the precession frequency of the rigid body.

IV. CONCLUSIONS

We have developed a geometric framework for the formulation of generalized Nambu systems. This formalism is more suitable from the view point of dynamical systems. As demonstrated with the example of Arter flow, a possibly non-integrable flow finds a desc ription in terms of generalized Nambu flow. Specifically, the Chandrashekhar flow and the Arter flow have been identified with the motion, which takes place in an invariant three dimensional subspace of a six dimensional Nambu system.

An interesting feature of the present formalism is the following. Whereas a three bracket of functions gives rise to a non-associative structure, a Nambu Poisson bracket of 2-forms gives rise to a Lie algebra. It was found that formulations involving 2-fo rms provide a natural approach to a Nambu system of order three. We feel that it is worth

investigating further the issues such as symmetries, reduction and integrability for such systems. Nambu systems of higher order could also be investigated. However, so far we have not carried this out, for the want of appropriate physical examples.

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APPENDIX

Definition IV.1 *(Block diagonal form) : Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold. A 2-form α is called block diagonal form if for some $X \in \mathcal{X}(M^{3n})$*

$$i_X \omega^{(3)} = \alpha$$

Definition IV.2 *(Non diagonal form) : Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold. A 2-form α is called non diagonal form if for each $z_1, z_2 \in \mathcal{X}(M^{3n})$ such that $\alpha(z_1, z_2) \neq 0$ $\not\exists$ X with the property $\omega^{(3)}(X, z_1, z_2) = \alpha(z_1, z_2)$.*

Remarks:

1. A form is not non diagonal form does not imply that it is block diagonal.

Proposition IV.1 *Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold, let α be any 2-form then*

$$\alpha = \alpha^d + \alpha'$$

where α^d is block diagonal form and α' is non diagonal form, and the decomposition is unique.

Proof: Consider Darboux coordinates

$$\begin{aligned}
\alpha &= \sum_{i,j=0}^{n-1} \sum_{l,m=1}^{3} \alpha_{3i+l \;\; 3j+m} dx^{3i+l} \wedge dx^{3j+m} \\
&= \sum_{i=0}^{n-1} \sum_{l,m=1}^{3} \alpha_{3i+l \;\; 3i+m} dx^{3i+l} \wedge dx^{3i+m} \\
&+ \sum_{i,j=0, i\neq j}^{n-1} \sum_{l,m=1}^{3} \alpha_{3i+l \;\; 3j+m} dx^{3i+l} \wedge dx^{3j+m}
\end{aligned}$$

It is easy to see that the first term which we denote by α^d is block diagonal form and the second term which we denote by α' is non diagonal form. Now we prove the uniqueness of the decomposition. Let α_1^d, α_1' and α_2^d, α_2' are two distinct decompositions of α such that α_1^d, α_2^d are block diagonal forms and α_1', α_2' are non diagonal forms. So we have $\alpha_1^d + \alpha_1' = \alpha_2^d + \alpha_2'$ this implies that $\alpha_2\prime$ is not non di agonal. hence α has unique decomposition.

$\square$

Definition IV.3 *We define a map*
$\sharp : \Omega_2^0(M^{3n}) \to \mathcal{X}(M^{3n}) : \alpha \mapsto \alpha^\sharp$ such that

$$i_{\alpha^\sharp} \omega^{(3)} = \alpha^d$$

where α^d is block diagonal part of α

This map can be identified as the map introduced in section II B. Since α^d is unique the definition of the map is coordinate free.

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